The Geography Problem and Constructing Exotic 4-Manifolds

Lecture 1

X, Y two 4-manifolds form an exotic pair if

X is homeomorphic to Y but

X is not diffeomorphic to Y

to find these we need 4 major steps

1. Examples
2. Constructions (tools to construct new manifolds)
3. Tools to prove homeomorphism
4. " " non-diffeo. (i.e. Invariants)

let X^n be an n-dimensional, compact, $\emptyset$ boundary, (closed)

smooth, orientable, and connected

$n=0$: point ✓

$n=1$: circle

$n=2$: ...

distinguish, π_1, H_1, H_*, $g \in \mathbb{N} \cup \{0\}$

$n=3$: Slogan: 3-mfds are determined by π_1

not strictly true!

if Y is sufficiently complicated then

$Y \# Y$ and $Y \# -Y$

have the same π_1 but can be very different

Poincaré conjecture: if $\pi_1(Y) = 1$, then $Y \cong S^3$

proved by Perelman

π_1 of 3-mfds are very special

$n = 4$:

Fact: if G is a finitely presented group then there exists X^4 such that $\pi_1(X) = G$

understanding groups is undecidable

so from now on assume $\pi_1(X) = 1$

Th^m (Freedman '82):

X_1, X_2 smooth, simply connected 4-manifolds are homeomorphic

easy $\Downarrow$ $\Longleftrightarrow$ $\Uparrow$ very hard

$H^*(X_1; \mathbb{Z}) \cong H^*(X_2; \mathbb{Z})$ as rings

need "as rings" e.g. $S^2 \times S^2$ and $\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$

have same cohomology as groups

$$H^*(X;\mathbb{Z}) = H^0(X) \oplus H^2(X) \oplus H^4(X)$$

$\uparrow$ $\pi_1 = 1$ 4-mfd; $H^0(X) \cong \mathbb{Z}$; $H^4(X) \cong \mathbb{Z}$

ring structure

$$H^2(X) \times H^2(X) \longrightarrow H^4(X)$$
$$(a, b) \longmapsto a \cup b$$

$$Q_X : H^2 \times H^2 \longrightarrow \mathbb{Z}$$
$$(a, b) \longmapsto (a \cup b, [X])$$

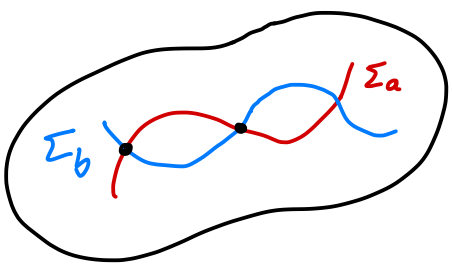
symmetric, unimodular form

(written as matrix determinant ± 1)

Q_X is the intersection form of X

$$Q_X(a,b) = \#(\Sigma_a \cap \Sigma_b)$$

Σ_a: homology P.D.(a); Σ_b: P.D.(b)



Homework: show for $a \in H^2(X)$, the Poincaré Dual of a can be realized by a smooth orientable surface

Three invariants of $Q_X = Q$

- rank of Q (= rank of free abelian group Q defined on)
- $\sigma(Q) = b_2^+(Q) - b_2^-(Q) \in \mathbb{Z}$
  ↖ signature
  where $b_2^\pm(Q)$ = # of ± 1 in diagonal after diagonalizing over $\mathbb{R}$
- Q is <u>even</u> if $Q(x,x)$ even $\forall x$
  <u>odd</u> otherwise

<u>Thm</u>: • if Q is <u>indefinite</u> (i.e. $\exists\, x, y$, $Q(x,x) > 0$, $Q(y,y) < 0$)
then $Q \sim n\langle 1\rangle \oplus m\langle -1\rangle$

$$= \begin{pmatrix} 1 & & & & 0 \\ & \ddots & & & \\ & & 1 & & \\ & & & -1 & \\ & & & & \ddots \\ 0 & & & & -1 \end{pmatrix}$$

(odd)
rank $= n+m$
$\sigma = n-m$

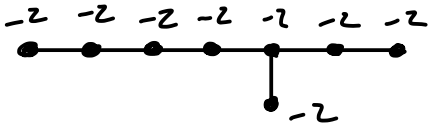
or

$Q \sim n E_8 \oplus \ell H$ $\quad \ell > 0$ (even)

$H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

rank $= 8|n| + 2|\ell|$
$\sigma = -8n$

E_8 8×8 specified by



- if Q is definite, for a fixed rank there are only finitely many different Q (but can be huge!) including $n\langle 1\rangle$ and $n\langle -1\rangle$

Thm: (Rokhlin, Donaldson, Furuta)

(R) X smooth, $\pi_1 = 1$, Q_X even (i.e. X spin) then $16 \mid \sigma(Q_X)$ (so coeff of E_8 is even)

(D) if Q_X is pos. definite for a smooth 4-manifold then $Q_X = n\langle 1\rangle$ similar for neg. definite

(D) if Q_X is even and $b_2^+(X) \leq 2$, then $Q_X \cong H$ or $2H$

(F) if Q_X is even, X smooth with $Q_X = 2n E_8 + \ell H$ then $\ell \geq 2|n|$ ($10/8$ theorem)

Conjecture: $\ell \geq 3|n|$ ($11/8$ conjecture)

↑ why? consider signature and Euler characteristic

Thm (Freedman, Donaldson):

X_1, X_2 two smooth, simply connected, closed 4-manifolds are homeomorphic

$\Leftrightarrow$

$b_2(X_1) = b_2(X_2)$ $(\Leftrightarrow \chi(X_1) = \chi(X_2))$

$\sigma(X_1) = \sigma(X_2)$

X_1 and X_2 are both even or both odd

Examples: If Q is odd $(\sim n\langle 1\rangle$ or $m\langle -1\rangle)$ then $\exists$ a 4-manifold X with $Q_X = Q$

$X = n\,\mathbb{CP}^2$ or $m\,\overline{\mathbb{CP}}^2$

If Q is even, $Q = 2n\,E_8 + \ell H$ then

$n\,K3 \# (\ell - 3n)\,S^2 \times S^2$ realizes Q

if $\ell \geq 3n$

$(Q_{K3} = 2E_8 \oplus 3H)$

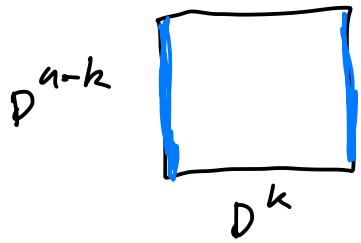
Homework:

- Show that $Q_{X_1 \# X_2} = Q_{X_1} \# Q_{X_2}$
- $E_8 \oplus \langle -1\rangle \not\cong 9\langle -1\rangle$ but same invariants

Constructions

Handle calculus

definition: an n-dimensional k-handle is $D^k \times D^{n-k}$ glued to an n-dimensional manifold X with boundary along $\partial D^k \times D^{n-k} = S^{k-1} \times D^{n-k}$



so in ∂X need an S^{k-1} with a trivialized normal bundle

trivialization are parameterized by $\pi_{k-1}(O(n-k))$

in dimension 4:

0-handle $D^0 \times D^4$ attached along $\partial D^0 \times D^4 = \emptyset$

- can always assume that for X connected (and closed) we can always assume we need a single 0-handle and single 4-handle
- the trivialization of the normal bundle is unique, except

when $k=2$ (here used X oriented)

in this case: $\pi_1(SO(3)) \cong \mathbb{Z}$

Fact: diffeomorphism of $\#_k S^1 \times S^2$ extends to a diffeomorphism of $\natural_k S^1 \times D^3$ (Laudenbach-Poenaru)

Fact: union of 0-handle and k 1-handles is $\natural_k S^1 \times D^3$

$=$ 4-handle and k 3-handles

So unique way to attach 3 and 4 handles

Suppose X is a 4-manifold with boundary built with one 0-handle, some 1-handles and some 2-handles

X can be symbolized as follows

$\mathbb{L}$ a link in S^3

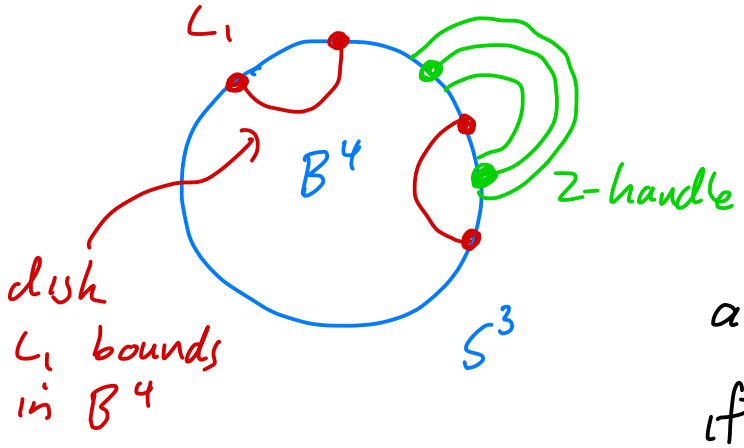
$\mathbb{L} = \mathbb{L}_1 \cup \mathbb{L}_2$

$\mathbb{L}_1$ = an ℓ-component unlink

put a dot on each

$\mathbb{L}_2$ = a link with each component equiped with an integer (i.e. framed

link)



$\mathbb{L}_1 = (L_1, \ldots, L_\ell)$

each L_i bounds disk in B^4

remove tubular neighborhood of disks

attach 2-handles to $\mathbb{L}_2$

if ∂ of result is $\#_k S^1 \times S^2$

then unique way to close it

Homework: X closed 4-manifold is given by $\mathbb{L}$ satisfying

- $\mathbb{L}_1 = \emptyset$
- $\partial(\mathbb{L}_2) = S^3$

then Q_X is just the linking matrix of $\mathbb{L}_2$ (with framings on the diagonal)